Journal of Applied Mechanics and Technical Physics, Vol. 40, No. 3, 1999

ABSOLUTE AND CONVECTIVE INSTABILITY OF A SUPERSONIC BOUNDARY LAYER

G. V. Petrov

UDC 532.526

Using the saddle-point method, asymptotics of time evolution for spatially localized three-dimensional intrinsic disturbances are determined. Criteria of absolute instability are established for the case of a branching dispersion relationship. Calculation results for the regions of existence of instability for a flat-plate boundary layer for $\mathrm{Re} \to \infty$ *and* $\mathrm{M} = 10$ *are presented.*

The term "absolute instability" implies the existence of spatially localized intrinsic linear disturbances, which cover the entire flow with time and infinitely grow at all points of the flow. An alternative notion of convective instability defines a class of flows, in which localized disturbances can reach a maximum value growing downstream, but leave a stabilizing flow behind them. Therefore, the growth of disturbances excited in a finite region is limited by the time of their passage through this region, whereas in the case of absolute instability it is limited by the time of existence of a given flow regime. Convectively unstable systems can only amplify the disturbances, whereas absolutely unstable systems can generate them [1].

The study of absolute instability in a free shear layer [2] shows that a rather strong counterflow is necessary for its existence; therefore, the instability of a two-dimensional boundary layer can be expected to have a convective character. Nevertheless, Petrov [3] found absolute instability of a supersonic boundary layer on a flat plate, which was caused by the existence of higher modes [4]. A unique dependence of frequency on the wavenumber (dispersion relationship) or its isolated branch were considered previously [1], and exactly the presence of branch points and their positions play a decisive role in the case of a supersonic boundary layer.

In the present paper, the results obtained by Petrov [3] are extended to three-dimensional disturbances in a flow depending only on the y coordinate. The integral

$$\hat{f}(x,y,z,t) = \int\limits_{\varphi_1}^{\varphi_2}\int\limits_{0}^{\infty} g(k,\varphi)\tilde{f}(k,\varphi,y)\exp\{i[k(x\cos\varphi + z\sin\varphi) - \omega(k,\varphi)t]\}\,dk\,d\varphi, \tag{1}$$

whose integrand is formed from the known solutions of the instability problem for oblique sinusoidal waves propagating at an angle φ to the direction of the Ox coordinate axis, is also the solution of this problem (superposition). Here i is the imaginary unity, x, y, and z are the coordinates (y is the distance from the wall), t is the time, $k = |\boldsymbol{k}|$ is the wavenumber, $\omega(k,\varphi)$ is a function that defines the dispersion relationship between the complex angular frequency $\omega = \omega_r + i\omega_i$ and the real wave vector $\boldsymbol{k}$, $\tilde{f}(k,\varphi,y)$ is the vector eigenfunction whose components are preexponents of the fluctuations of velocity, pressure, and other flow parameters, and the function $g(k,\varphi)$ is arbitrary. If $g(k,\varphi)$ does not contain δ-functions and integral (1) converges in the classical sense, then $\hat{f}(x,y,z,t)$ is a spatially localized intrinsic disturbance.

The internal integral in Eq. (1) for a given φ is a two-dimensional disturbance localized in the direction of its propagation. The use of the saddle-point method [5] for $t \to \infty$ yields the asymptotic formula

$$\hat{f}(x,y,z,t) \sim t^{-1/2}\int\limits_{\varphi_1}^{\varphi_2} g_s(\varphi)f_s(\varphi,y)\exp\{i[k_s(\varphi)(x\cos\varphi + z\sin\varphi) - \omega_s(\varphi)t]\}\,d\varphi, \tag{2}$$

Institute of Theoretical and Applied Mechanics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 40, No. 3, pp. 104–108, May–June, 1999. Original article submitted June 19, 1998.

0021-8944/99/4003-0451 $22.00

where $g_s(\varphi) = \sqrt{2\pi/i\lambda(\varphi)}\, g(k_s(\varphi),\varphi)$, $\lambda(\varphi) = \partial^2\omega(k,\varphi)/\partial k^2$ for $k = k_s(\varphi)$, $f_s(\varphi, y) = \tilde{f}(k_s(\varphi),\varphi,y)$, $\omega_s(\varphi) = \omega(k_s(\varphi),\varphi)$, $k_s(\varphi)$ is the saddle point defined as the solution of the equation $\partial\omega(k,\varphi)/\partial k = 0$ extended to the complex plane $k = k_r + ik_i$.

If there exists a saddle point φ_σ of the function $\omega_s(\varphi)$, which is defined as the solution of the equation $d\omega_s/d\varphi = 0$, its contribution to the asymptotic can be evaluated by repeated use of the saddle-point method for Eq. (2):

$$\hat{f}(x,y,z,t) \sim t^{-1}\sqrt{2\pi/i\omega_s''(\varphi_\sigma)}\, g_\sigma f_\sigma(y) \exp\{i[k_\sigma(x\cos\varphi_\sigma + z\sin\varphi_\sigma) - \omega_\sigma t]\}, \quad t \to \infty, \tag{3}$$

where $g_\sigma = g_s(\varphi_\sigma)$, $f_\sigma(y) = f_s(\varphi_\sigma, y)$, $k_\sigma = k_s(\varphi_\sigma)$, $\omega_\sigma = \omega_s(\varphi_\sigma)$, and the primed symbols are derivatives.

The contribution of the integration boundaries φ_b, i.e., φ_1, φ_2, etc. [see (1) and (2)], to the asymptotics is determined by another formula

$$\hat{f}(x,y,z,t) \sim t^{-3/2}[g_b/i\omega_s'(\varphi_b)] f_b(y) \exp\{i[k_b(x\cos\varphi_b + z\sin\varphi_b) - \omega_b t]\} \quad (t \to \infty), \tag{4}$$

where $g_b = g_s(\varphi_b)$, $f_b(y) = f_s(\varphi_b, y)$, $k_b = k_s(\varphi_b)$, and $\omega_b = \omega_s(\varphi_b)$.

Numerical studies are performed for a flat-plate boundary layer for $\mathrm{Re} \to \infty$. The equations of inviscid theory and the boundary conditions for oblique elementary waves have the following form [4]:

$$\left.\begin{aligned} \tilde{v}' &= \frac{u'}{u-c}\tilde{v} + \Big(\frac{1}{s} - u + c\Big) ik\tilde{p}, \\ \tilde{p}' &= -iks\tilde{v}, \end{aligned}\right\} \quad s = \gamma \mathrm{M}^2 \rho(u-c)\cos^2\varphi, \quad c = \omega/k,$$

$$\tilde{v} \to 0, \quad \tilde{p} \to 0 \quad \text{for} \quad y \to \infty, \qquad \tilde{v} = 0 \quad \text{for} \quad y = 0,$$

where $\tilde{p}$ and $\tilde{v}$ are the fluctuations of pressure and normal-to-wall component of velocity, the primed symbols are derivatives relative to y, ρ and u are the density and velocity profiles, γ is the ratio of specific heats in a gas, and M is the Mach number. In calculating the boundary-layer profiles, it is assumed that the gas is perfect, the viscosity is proportional to temperature, and the Prandtl number is Pr=1.

Petrov [3] studied two-dimensional instability for $\varphi = 0$. It is shown that there is a saddle point k_s in the vicinity of each branch point k_b of the function $\omega(k)$ and the branch point itself is a saddle point of the inverse function $k(\omega)$. A similar symmetry is the typical feature of the existence conditions for absolute instability: the known requirement of the growth in time of $\omega_{si} > 0$ (the subscript i indicates the imaginary part) of an elementary wave corresponding to the saddle point is supplemented by the requirement of spatial growth of $k_{bi} < 0$ in the direction $\boldsymbol{k}$ of the wave corresponding to the branch point, i.e.,

$$\omega_i > 0 \quad \text{for} \quad \frac{\partial\omega}{\partial k} = 0, \qquad k_i < 0 \quad \text{for} \quad \frac{\partial k}{\partial\omega} = 0. \tag{5}$$

Two types of absolute instability were found: the first type is caused by branching of subsonic modes with each other, the second type is caused by branching of subsonic and supersonic modes.

Generalizing the results obtained to oblique waves in the vicinity of the angle $\varphi = 0$, we introduce the following notation for the boundaries of two-dimensional absolute instability:

$$B_{mn}: \quad \frac{\partial k}{\partial\omega} = 0, \quad k_i = 0, \qquad S_{mn}: \quad \frac{\partial\omega}{\partial k} = 0, \quad \omega_i = 0,$$

where m and n are the numbers of branching modes. In the general case, they are surfaces in the space of the flow parameters and the angle φ of propagation of an elementary wave.

Figure 1 shows the regions of two-dimensional absolute instability (5) calculated for $\mathrm{M} = 10$ and $\gamma = 1.4$. This instability is caused by branching of subsonic modes: region II is formed by branching of the second and third modes and region III by branching of the third and fourth modes (T_w is the wall temperature relative to the flow temperature). In these and other figures, the boundaries B_{mn} are plotted by solid curves and the boundaries S_{mn} by dot-and-dashed curves. The consideration of oblique waves expands the range of wall temperatures for which two-dimensional absolute instability exists [3].

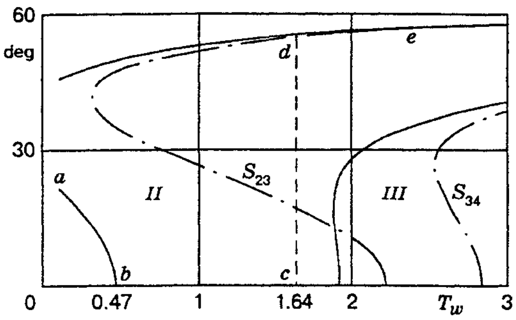

Fig. 1

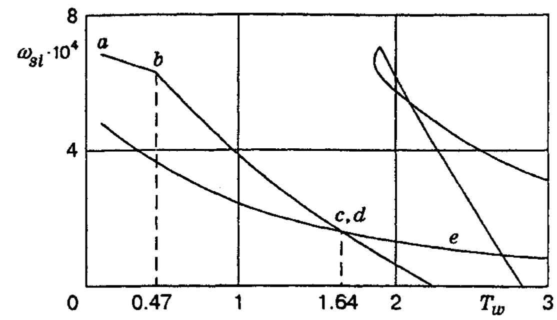

Fig. 2

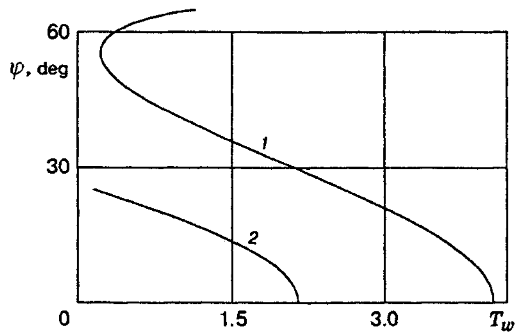

Fig. 3

For planar flows, the disturbance characteristics are even functions of the angle φ; therefore, the point $\varphi = 0$ is a saddle point of the function $\omega_s(\varphi)$. Judging by the configuration of the dot-and-dashed curves S_{23} and S_{34}, we can tell that this is the point of the maximum growth rate of $\omega_{si}(\varphi)$ (this fact is validated by direct calculations). Thus, formula (3) is valid, from which it follows that three-dimensional absolute instability also exists in the regions of existence of two-dimensional absolute instability calculated by Petrov [3]. Small initial three-dimensional perturbations excite a wave with an equal frequency, length, y-distribution of the amplitude, and even asymptotic growth rate, since the power factor in time is insignificant for large t.

As in the planar problem, there are no asymptotics for oblique localized disturbances determined by the integrand in Eq. (2) outside the boundaries B_{mn} of the corresponding regions of absolute instability, since the path of integration of Eq. (1) with respect to k from one valley to another through a saddle point leads to another branch and is not equivalent to the real semi-axis. Hence, B_{mn} determine the boundaries φ_b of integration in relation (2), and their contribution to the asymptotics of a three-dimensional localized disturbance is described by formula (4). Because of the symmetry, these are two oblique waves propagating at angles $\varphi = \pm\varphi_b$ (they are plotted by solid curves in Figs. 1, 3, and 4).

The growth rates $\omega_{si}(\varphi_b)$ and $\omega_{si}(0)$, which determine the asymptotic growth rate of disturbances in time, are shown in Fig. 2. They are normalized using the time scale $\sqrt{\nu_e x/u_e^3}$, where ν_e and u_e are the kinematic viscosity and the free-stream velocity and x is the distance from the plate edge. For $T_w = 0.47$, the highest growth rates correspond to waves with a moderate angle of propagation $\varphi_b \cong 20°$–0 (section ab in Figs. 1 and 2). Within the range $T_w = 0.47$–1.64 (section bc in Fig. 1), the asymptotic is a plane wave propagating along the flow. For $T_w > 1.64$ (section de in Figs. 1 and 2), the asymptotic is an oblique wave with a large angle of propagation ($\varphi_b > 55°$).

As the wall temperature is further increased, absolute instability corresponding to branching of the third and fourth modes appears and dominates beginning from $T_w = 1.88$, and the sequence of asymptotics is repeated.

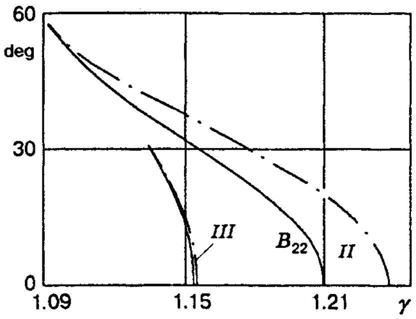

Fig. 4

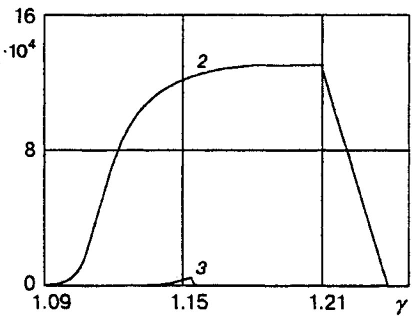

Fig. 5

Figure 3 shows the boundaries B_{12} of the regions of absolute instability caused by branching of the first and second subsonic modes for M = 8.3 (curve 1) and M = 10 (curve 2). With increasing Mach number, the boundary moves toward the lower temperatures of the wall. The calculations for higher neighboring modes yield an identical result.

Absolute instability of the second kind, which is cause by branching of subsonic and supersonic modes, occurs for lower values of the ratio of specific heats γ. The existence regions of this instability for M = 10 and $T_w = 1$ are plotted in Fig. 4. Region II corresponds to the second mode and region III to the third mode. The notation in Fig. 5, which shows the asymptotic growth rate versus γ, is also associated with the mode numbers. Absolute instability for the disturbances of the third mode is very weakly expressed. Concerning the second mode, only oblique waves with the growth rate $\omega_{si}(\varphi_b)$ (φ_b is determined by the boundary B_{22} of region II in Fig. 4) can be self-excited for $\gamma < 1.210$ and a streamwise wave with the growth rate $\omega_{si}(0)$ is excited within the range $\gamma = 1.210 - 1.238$.

Thus, the two-dimensional absolute instability found by Petrov [3] corresponds also to three-dimensional absolute instability within the same range of the boundary-layer parameters and with the same asymptotic wave characteristics of the disturbances. In addition, there is an asymptotic of three-dimensional localized disturbances in the form of oblique sinusoidal waves growing with time, which significantly expands the range of the existence parameters of absolute instability. The angles of propagation of these waves are determined by the branch points of the function $\omega(\boldsymbol{k})$ on the real plane $\boldsymbol{k}$.

REFERENCES

1. A. M. Fedorchenko and N. Ya. Kotsarenko, *Absolute and Convective Instability in Plasma and Solids* [in Russian], Nauka, Moscow (1981).
2. I. S. Shikina, "Asymptotics of localized perturbations in free shear layers," *Mekh. Zhidk. Gaza*, No. 2, 8–14 (1987).
3. G. V. Petrov, "Two-dimensional absolute instability of a supersonic boundary layer," *Mekh. Zhidk. Gaza*, No. 1, 176–179 (1988).
4. L. M. Mack, "Boundary layer stability theory," JPL Document No. 900–277 (Rev. A), JPL, Pasadena (1969).
5. A. Erdelyi, *Asymptotic Expansions*, Dover Publications, Inc., New York, (1956).